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# The Effects of Hyperbolic Eigenparameter on Spectral Analysis of a Quantum Difference Equations

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## ABSTRACT

In this study, second-order nonselfadjoint expression and its associated boundary condition depending on an hyperbolic eigenparameter are discussed. We introduce the sets of eigenvalues and spectral singularities of a boundary value problem (BVP) which is defined with same quantum difference expression and boundary condition. Next, some spectral properties of eigenvalues and spectral singularities are investigated using the Jost solution, green function and resolvent operator of this BVP.

**Keywords:** Eigenvalue, spectral singularity, $q$-difference equation, green function, Jost solution, resolvent.

# 1. Introduction

Time scale theory was first presented by Hilger (1988) in his doctoral dissertation. Dynamic equations on time scales have been introduced by Hilger (1990) to extend the theory of ordinary differential, difference and quantum equations which are based on the $q$-calculus and the $h$-calculus (Kac and Cheung (2002)), defined over non-empty closed subsets of the real line. Several important problems concerning higher-order dynamic equations on arbitrary time scales which are general cases of $q$-difference equations in Bohner et al. (2003) and Bohner and Peterson (2001). In Gulsen and Yilmaz (2016), authors investigate the spectral theory of Dirac system on time scales. Quantum difference equations have huge applications in several disciplines such as physics, mathematics, biology, engineering and economics.

In this way, mathematicians begin to deal with the topic of the spectral analysis of $q$-difference equations and operators. In the work done by Adivar and Bohner (2006), the spectral analysis of nonselfadjoint $q$-difference equations have been investigated Adivar and Bohner (2006). In last two years, the spectral analysis of $q$-difference equations and operators both including a polynomial type Jost solution and exponential type Jost solution in ordinary and matrix cases have been studied in Aygar (2016a), Aygar and Bohner (2015) and Aygar and Bohner (2016b). Also, BVPs related with $q$ -difference equations have been considered in Aygar (2015), Aygar (2016c) and Aygar and Bohner (2016a).

This study is different from the studies which are mentioned above, in this work, we use hyperbolic eigenparameter to get the spectral analysis of a $q$-difference equation. We extend the results given in Aygar (2016b) by getting important properties of eigenvalues and spectral singularities of this $q$-difference equation.

To obtain our main results about eigenvalues and spectral singularities, we use properties of Jost solution, green function and resolvent given in Aygar (2016b). At the end, it can be seen that the differences and similarities of using trigonometric eigenparameter (Adivar and Bohner (2006), Aygar (2015), Aygar (2016a), Aygar and Bohner (2016a), Bohner and Koyunbakan (2016)) and using hyperbolic eigenparameter on the spectral analysis of these difference equations.

As a result of using hyperbolic eigenparameter, analytical region of Jost solution and the region of eigenvalues and spectral singularities has changed. In present paper, we use the notation $q^{\mathbb{N}_0} := \{q^n : n \in \mathbb{N}_0\}$ for $q > 1$, where

$\mathbb{N}_0$ denotes the set of nonnegative integers. We use a Hilbert space $\ell_2(q^{\mathbb{N}})$ of complex-valued functions defining with the inner product

$$\langle f, g\rangle_q := \sum_{t\in q^{\mathbb{N}_0}} \mu(t) f(t)\overline{g(t)}, \quad f, g : q^{\mathbb{N}_0} \to \mathbb{C}$$

and the norm

$$\|f\|_q := \left(\sum_{t\in q^{\mathbb{N}_0}} \mu(t)|f(t)|^2\right)^{\frac{1}{2}}, \quad f : q^{\mathbb{N}_0} \to \mathbb{C}$$

where $\mu(t) = (q-1)t$ for all $t \in q^{\mathbb{N}_0}$.

Let us consider the nonselfadjoint BVP

$$qa(t)y(qt) + b(t)y(t) + a\left(\frac{t}{q}\right)y\left(\frac{t}{q}\right) = \lambda y(t), \quad t \in q^{\mathbb{N}} \tag{1}$$

and the boundary condition

$$\begin{gathered}(\gamma_0 + \gamma_1\lambda)y(q) + (\beta_0 + \beta_1\lambda)y(1) = 0, \\ \gamma_0\beta_1 - \gamma_1\beta_0 \neq 0, \quad \gamma_1 \neq \tfrac{\beta_0}{a(1)},\end{gathered} \tag{2}$$

where $\{a(t)\}_{t\in q^{\mathbb{N}_0}}$ and $\{b(t)\}_{t\in q^{\mathbb{N}}}$ are complex sequences, $\lambda$ is an eigenparameter, $a(t) \neq 0$ for all $t \in q^{\mathbb{N}_0}$, and $\gamma_i, \beta_i \in \mathbb{C}$, $i = 0, 1$.

The paper is organized as follows: In Section 2, essential results which are given by Aygar (2016b) are included for the convenience of the reader. Due to the erratum of Aygar (2016b), here there are some small differences in the aspect of green function and the proof of Theorem 1 in Aygar (2016b).

In Section 3, we introduce the sets of eigenvalues and spectral singularities of the BVP (1)–(2) with hyperbolic eigenparameter and we get the properties of these sets under the condition

$$\sum_{t\in q^{\mathbb{N}}} (|1 - a(t)| + |b(t)|) < \infty. \tag{3}$$

## 2. Preliminaries

Under the condition (3), the solution of BVP (1)–(2) is given

$$e(t,z)=\alpha(t)\frac{e^{\frac{\ln t}{\ln q}z}}{\sqrt{\mu(t)}}\left(1+\sum_{r\in q^{\mathbb{N}}}A(t,r)e^{\frac{\ln r}{\ln q}z}\right),t\in q^{\mathbb{N}_0} \tag{4}$$

for $\lambda=2\sqrt{q}\cosh z$ in Aygar (2016b), where $z\in\overline{\mathbb{C}}_-:=\{z\in\mathbb{C}:\operatorname{Re}z\leq 0\}$, $\mu(t)=(q-1)t$ for all $t\in q^{\mathbb{N}_0}$ and $\alpha(t)$, $A(t,r)$ are expressed in terms of $\{a(t)\}$ and $\{b(t)\}$. In Aygar (2016b), the author shows that $A(t,r)$ satisfies

$$|A(t,r)|\leq C\sum_{s\in\left[tq^{\lfloor\frac{\ln r}{2\ln q}\rfloor},\infty\right)\cap q^{\mathbb{N}}}\left(|1-a(s)|+|b(s)|\right), \tag{5}$$

where $\lfloor\frac{\ln r}{2\ln q}\rfloor$ is the integer part of $\frac{\ln r}{2\ln q}$ and $C>0$ is a constant. Using (3) it is seen that $e(\cdot,z)$ is analytic with respect to $z$ in $\mathbb{C}_-:=\{z\in\mathbb{C}:\operatorname{Re}z<0\}$ and continuous in $\overline{\mathbb{C}}_-$.

Using (4) and the boundary condition (2), we define the function $g$ by

$$\begin{aligned} g(z) \quad = \quad & (\gamma_0+2\sqrt{q}\gamma_1\cosh z)e(q,z) \\ & +(\beta_0+2\sqrt{q}\beta_1\cosh z)e(1,z). \end{aligned} \tag{6}$$

The function $g$ is analytic in $\mathbb{C}_-$, continuous in $\overline{\mathbb{C}}_-$, and $g(z)=g(z+2i\pi)$. Analogously to the Sturm–Liouville differential equation, the solution $e(\cdot,z)$ and the function $g$ are called the Jost solution and Jost function of (1)–(2), respectively (Naĭmark (1968)). Let $\varphi(\lambda)=\{\varphi(t,z)\}$ $t\in q^{\mathbb{N}_0}$, be the solution of (1) satisfying the initial conditions

$$\varphi(1,\lambda)=-(\gamma_0+\gamma_1\lambda),\quad \varphi(q,\lambda)=(\beta_0+\beta_1\lambda).$$

If we define

$$\phi(t,z)=\varphi(2\sqrt{q}\cosh z)=\{\varphi(t,2\sqrt{q}\cosh z)\}_{t\in q^{\mathbb{N}_0}},$$

then $\phi$ is an entire function and $\phi(z)=\phi(z+2i\pi)$. Let us take the semi-strips $P_0=\{z\in\mathbb{C}_-:-\frac{\pi}{2}\leq\operatorname{Im}z\leq\frac{3\pi}{2}\}$ and $P=P_0\cup\left[-\frac{i\pi}{2},\frac{3i\pi}{2}\right]$. The Wronskian of two solutions $y=\{y(t,\lambda)\}_{t\in q^{\mathbb{N}}}$ and $v=\{(t,\lambda)\}_{t\in q^{\mathbb{N}}}$ of (1) is defined by

$$W[y,v]=\mu(t)a(t)\{y(t,\lambda)v(qt,\lambda)-y(qt,\lambda)v(t,\lambda),\quad t\in q^{\mathbb{N}_0}.$$

We find $W[\phi(t,z), e(t,z)] = -\mu(1)a(1)g(z)$. Since $\phi(\cdot,z)$ and $e(\cdot,z)$ are independent solutions for all $z \in P$ with $g(z) \neq 0$, we find the Green function of the BVP (1)– (2)

$$G_{t,r}(z) := \begin{cases} -\frac{\phi(r,z)e(t,z)}{qa(1)g(z)}, & r = tq^{-k}, \quad k \in \mathbb{N}_0 \\ -\frac{e(r,z)\phi(t,z)}{qa(1)g(z)}, & r = tq^{k}, \quad k \in \mathbb{N}. \end{cases} \tag{7}$$

It is obvious that

$$(Rh)(t) := \sum_{r \in q^{\mathbb{N}}} G(t,r)h(r), \quad h \in \ell_2(q^{\mathbb{N}}) \tag{8}$$

is the resolvent of the BVP (1)–(2). Related to the equation (1), we will recall the $q$-difference operator by $L$ generated in $\ell_2(q^{\mathbb{N}})$ by the $q$-difference expression

$$(ly)(t) := a\left(\frac{t}{q}\right)y\left(\frac{t}{q}\right) + b(t)y(t) + qa(t)y(qt), \quad t \in q^{\mathbb{N}}.$$

Let $L_1$ and $L_2$ denote the $q$-difference operators generated in $\ell_2(q^{\mathbb{N}})$ by

$$(l_1y)(t) := y\left(\frac{t}{q}\right) + qy(qt), \quad t \in q^{\mathbb{N}}$$

and

$$(l_2y)(t) := \left(a\left(\frac{t}{q}\right) - 1\right)y\left(\frac{t}{q}\right) + b(t)y(t) + q(a(t)-1)y(qt), \quad t \in q^{\mathbb{N}},$$

respectively. It is clear that $L_1$ is a self-adjoint operator, $L_2$ is a compact operator in $\ell_2(q^{\mathbb{N}})$ and $L = L_1 + L_2$ (Aygar and Bohner (2015)). For all $z \in \mathbb{C}_- \setminus \{ik\pi : k \in \mathbb{Z}\}$, we define the green function and resolvent operator of $L_1$ by

$$S_{t,r}(z) := \frac{\sqrt{q}}{2sinhz} \begin{cases} \frac{e^{-\frac{\ln r}{\ln q}z}}{\sqrt{\mu(r)}}\frac{e^{\frac{\ln t}{\ln q}z}}{\sqrt{\mu(t)}}, & r = tq^{-k}, \quad k \in \mathbb{N} \\ \frac{e^{\frac{\ln r}{\ln q}z}}{\sqrt{\mu(r)}}\frac{e^{-\frac{\ln t}{\ln q}z}}{\sqrt{\mu(t)}}, & r = tq^{k}, \quad k \in \mathbb{N}_0. \end{cases} \tag{9}$$

and

$$R_\lambda(L_1)\psi(t) := \sum_{r \in q^{\mathbb{N}}} S(t,r)\psi(r), \quad \psi \in \ell_2(q^{\mathbb{N}}) \tag{10}$$

respectively (Aygar (2016b)). Moreover in Aygar (2016b), it is seen that for every $\delta > 0$, there is a number such that $C_\delta$ such that

$$\|R_\lambda(L_1)\|_q > \frac{C_\delta}{|sinhz|\sqrt{1-e^{-2\operatorname{Re} z}}}.$$

Using last inequality and assumption (2), it is shown that

$$\sigma(L_1) = \sigma_{\mathrm{c}}(L_1) = \sigma_{\mathrm{c}}(L) = [-2\sqrt{q}, 2\sqrt{q}],$$

where $\sigma(L_1)$ and $\sigma_{\mathrm{c}}(L_1)$ denote the spectrum and continuous spectrum of the operator $L_1$, respectively.

## 3. Properties of Eigenvalues and Spectral Singularities of L

By using (9), (10) and the definitions of the eigenvalues and the spectral singularities given in Naĭmark (1968), we can write

$$\sigma_d = \{\lambda \in \mathbb{C} : \lambda = 2\sqrt{q}\cosh z,\ z \in P_0,\ g(z) = 0\}, \tag{11}$$

$$\sigma_{ss} = \left\{\lambda \in \mathbb{C} : \lambda = 2\sqrt{q}\cosh z, z = i\tau,\ z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right],\ g(z) = 0\right\} \setminus \{0\}. \tag{12}$$

Using (4) and (6), we get

$$\begin{aligned} g(z) \;=\; & \alpha(1)\sqrt{\frac{q}{q-1}}\beta_1 e^{-z} + \alpha(q)\frac{\gamma_1}{\sqrt{q-1}} + \alpha(1)\frac{\beta_0}{\sqrt{q-1}} \\ & + \left(\alpha(q)\frac{\gamma_0}{\sqrt{q(q-1)}} + \alpha(1)\sqrt{\frac{q}{q-1}}\beta_1\right) e^{z} \\ & + \alpha(q)\frac{\gamma_1}{\sqrt{q-1}} e^{2z} + \sum_{r \in q^{\mathbb{N}}} \alpha(1)\sqrt{\frac{q}{q-1}}\beta_1 A(1,r) e^{\left(\frac{\ln r}{\ln q}-1\right)z} \\ & + \sum_{r \in q^{\mathbb{N}}} \left(\alpha(q)\frac{\gamma_1}{\sqrt{q-1}}A(q,r) + \alpha(1)\frac{\beta_0}{\sqrt{q-1}}A(1,r)\right) e^{\frac{\ln r}{\ln q}z} \\ & + \sum_{r \in q^{\mathbb{N}}} \left(\alpha(q)\frac{\gamma_0}{\sqrt{q(q-1)}}A(q,r) + \alpha(1)\sqrt{\frac{q}{q-1}}\beta_1 A(1,r)\right) e^{\left(\frac{\ln r}{\ln q}+1\right)z} \\ & + \sum_{r \in q^{\mathbb{N}}} \alpha(q)\frac{\gamma_1}{\sqrt{q-1}}A(q,r) e^{\left(\frac{\ln r}{\ln q}+2\right)z}. \end{aligned}$$

Let us define

$$F(z) := g(z)e^{z}, \tag{13}$$

then we can write the function $F$ as

$$\begin{aligned}
F(z) \;=\;& \alpha(1)\sqrt{\frac{q}{q-1}}\beta_1 + \left(\alpha(q)\frac{\gamma_1}{\sqrt{q-1}} + \alpha(1)\frac{\beta_0}{\sqrt{q-1}}\right)e^{z} \\
&+ \left(\alpha(q)\frac{\gamma_0}{\sqrt{q(q-1)}} + \alpha(1)\sqrt{\frac{q}{q-1}}\beta_1\right)e^{2z} \\
&+\alpha(q)\frac{\gamma_1}{\sqrt{q-1}}e^{3z} + \sum_{r\in q^{\mathbb{N}}}\alpha(1)\sqrt{\frac{q}{q-1}}\beta_1 A(1,r)e^{\frac{\ln r}{\ln q}z} \\
&+ \sum_{r\in q^{\mathbb{N}}}\left(\alpha(q)\frac{\gamma_1}{\sqrt{q-1}}A(q,r)\right. \\
&\qquad\left. +\alpha(1)\frac{\beta_0}{\sqrt{q-1}}A(1,r)\right)e^{\left(\frac{\ln r}{\ln q}+1\right)z} \\
&+ \sum_{r\in q^{\mathbb{N}}}\left(\alpha(q)\frac{\gamma_0}{\sqrt{q(q-1)}}A(q,r)\right. \\
&\qquad\left. +\alpha(1)\sqrt{\frac{q}{q-1}}\beta_1 A(1,r)\right)e^{\left(\frac{\ln r}{\ln q}+2\right)z} \\
&+ \sum_{r\in q^{\mathbb{N}}}\alpha(q)\frac{\gamma_1}{\sqrt{q-1}}A(q,r)e^{\left(\frac{\ln r}{\ln q}+3\right)z}.
\end{aligned} \tag{14}$$

From the continuity, analyticity of the function $g$, we find that the function $F$ is analytic in $\mathbb{C}_-$, continuous in $\overline{\mathbb{C}}_-$. Since the function $g$ satisfies $g(z) = g(z+2i\pi)$, we can also write $F(z) = F(z+2i\pi)$. It follows from (11)–(13) that

$$\sigma_d = \{\lambda = 2\sqrt{q}\cosh z : z \in P_0,\ F(z) = 0\}, \tag{15}$$

$$\sigma_{ss} = \left\{\lambda = 2\sqrt{q}\cosh z : z = i\tau, \tau \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right], F(z) = 0\right\} \setminus \{0\}. \tag{16}$$

Now, we will assume a stronger condition than (4) as

$$\sum_{t\in q^{\mathbb{N}}} \frac{\ln t}{\ln q}(|1-a(t)| + |b(t)|) < \infty. \tag{17}$$

**Theorem 3.1.** *Assume* (17)*. Then for all $z \in P$, the function $F$ satisfies*

$$F(z) = \frac{\sqrt{q}}{\sqrt{q-1}}\beta_1\alpha(1) + O(e^{\operatorname{Re} z}), \quad \beta_1 \neq 0, \quad \operatorname{Re} z \to -\infty, \tag{18}$$

$$F(z) = \frac{1}{\sqrt{q-1}}\left[\gamma_1\alpha(q) + \beta_0\alpha(1)\right] e^{z} + O(e^{2\operatorname{Re} z}), \quad \beta_1 = 0, \quad \operatorname{Re} z \to -\infty. \tag{19}$$

*Proof.* Using (5), we get

$$\sum_{r\in q^{\mathbb{N}}} |A(t,r)| e^{\frac{\ln r}{\ln q}\operatorname{Re} z} \leq 2Ce^{\operatorname{Re} z} \sum_{p\in[t,\infty)\cap q^{\mathbb{N}}} \frac{\ln p}{\ln t}(|1-a(t)| + |b(t)|). \tag{20}$$

If $\operatorname{Re} z \to -\infty$, we get

$$2Ce^{\operatorname{Re} z} \sum_{p\in[t,\infty)\cap q^{\mathbb{N}}} \frac{\ln p}{\ln t}(|1-a(t)| + |b(t)|) \to 0,$$

where $C$ is a positive constant.

Also, we can write

$$\sum_{r\in q^{\mathbb{N}}} |A(t,r)| e^{(\frac{\ln r}{\ln q}+m)\operatorname{Re} z} \leq \sum_{r\in q^{\mathbb{N}}} |A(t,r)| e^{\frac{\ln r}{\ln q}\operatorname{Re} z} \to 0, \quad \operatorname{Re} z \to -\infty \tag{21}$$

for $m = 1, 2, 3$ and $z \in P$. From (14), (20) and (23), there is a positive real number $M$ which satisfies

$$\left| \frac{F(z) - \frac{\sqrt{q}\alpha(1)\beta_1}{\sqrt{\mu(1)}}}{e^{\operatorname{Re} z}} \right| \leq M$$

for $\beta_1 \neq 0$, $z \in P$, $\operatorname{Re} z \to -\infty$. Similarly, for $\beta_1 = 0$, (19) can be found by using (14) and (20). □

**Definition 3.1.** *The multiplicity of a zero of $F$ in $P$ is called the multiplicity of the corresponding eigenvalue or spectral singularity of BVP* (1)–(2)*.*

Let us define

$$\begin{array}{lll} M_1 & := & \left\{z \in P_0 : \ F(z) = 0\right\}, \\ M_2 & := & \left\{z = i\tau : z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] : \ F(z) = 0\right\}. \end{array} \quad (22)$$

We also denote the set of all limit points of $M_1$ by $M_3$ and the set of all zeros of $F$ with infinite multiplicity in $P$ by $M_4$. From (15), (16) and (22), we get that

$$\begin{array}{lll} \sigma_d & = & \left\{\lambda = 2\sqrt{q}\cosh z, \ z \in M_1\right\}, \\ \sigma_{ss} & = & \left\{\lambda = 2\sqrt{q}\cosh z, \ z \in M_2\right\} \setminus \{0\}. \end{array} \quad (23)$$

**Theorem 3.2.** *Under the assumption* (17)*, we get:*

*i)* *The set $M_1$ is bounded and countable.*

*ii)* $M_1 \cap M_3 = \emptyset$, $M_1 \cap M_4 = \emptyset$.

*iii)* *The set $M_2$ is compact and the Lebesgue measure of $M_2$ in the imaginary axis is zero.*

*iv)* $M_3 \subset M_2$, $M_4 \subset M_2$, *the Lebesgue measure of $M_3$ and $M_4$ are also zero.*

*v)* $M_3 \subset M_4$.

*Proof.* From (18) and (19), we find the boundedness of the set $M_1$. Since $F$ is a $2i\pi$-periodic function and is analytic in $\mathbb{C}_-$, we get that $M_1$ has at most a countable number of elements. We also find (i)–(iv) easily from the boundary uniqueness theorem of analytic functions (Dolzhenko (2016)). Also, (v) is obtained using the continuity of all derivatives of $F$ on $\left[-i\frac{\pi}{2}, i\frac{3\pi}{2}\right]$. $\square$

As a result of Theorem 3.1 and (23), we obtain the following:

**Remark 3.1.** *Under the condition* (17)*, the set $\sigma_d$ is bounded, has at most countable number of elements and its limit points can lie only in $[-2\sqrt{q}, 2\sqrt{q}]$. Also $\sigma_{ss} \subset [-2\sqrt{q}, 2\sqrt{q}]$ and the Lebesgue measure of the set $\sigma_{ss}$ in the imaginary axis is zero.*

Now, we will use a different condition to investigate the quantitative properties of the sets $\sigma_d$ and $\sigma_{ss}$. Let us suppose that complex sequences $\{a(t)\}_{t\in q^{\mathbb{N}_0}}$ and $\{b(t)\}_{t\in q^{\mathbb{N}}}$ satisfy

$$\sup_{t\in q^{\mathbb{N}}}\left\{\exp\left[\varepsilon\left(\tfrac{\ln t}{\ln q}\right)^{\delta}\right](|1-a(t)|+|b(t)|)\right\}<\infty, \qquad (24)$$
$$\varepsilon>0, \quad \tfrac{1}{2}\leq\delta<1.$$

It is clear that we obtain

$$\sup_{t\in q^{\mathbb{N}}}\left\{\exp\left(\varepsilon\frac{\ln t}{\ln q}\right)(|1-a(t)|+|b(t)|)\right\}<\infty, \quad \varepsilon>0, \qquad (25)$$

for $\delta=1$.

**Theorem 3.3.** *Assume* (25). *Then the BVP* (1)–(2) *has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity.*

*Proof.* Using (5) and (25), we obtain

$$|A(t,r)|\leq C_1\exp\left[-\frac{\varepsilon}{2}\left(\frac{\ln r}{\ln q}+\frac{\ln r}{2\ln q}\right)\right], \ t\in\{1,q\}, \quad r\in q^{\mathbb{N}}, \qquad (26)$$

where $C_1>0$ is a constant. It follows from last inequality and (20) that the function $F$ has an analytic continuation to the right half-plane $\operatorname{Re} z<\frac{\varepsilon}{4}$. Since $F$ is a $2i\pi$ periodic function, the limit points of zeros of $P$ can not lie on the boundary of $P$. By using Theorem 3.1, we obtain that the bounded sets $M_1$ and $M_2$ have a finite number of elements. From the analyticity of function $F$ in $\operatorname{Re} z<\frac{\varepsilon}{4}$, we get that all zeros of $F$ in $P$ have finite multiplicity. □

As you see above, (25) is stronger condition than (24). Now, we will give the same result as Theorem 3.3 under the condition (24) by using different way of proof. Because under the condition (24), the function $F$ does not have an analytic continuation from the imaginary axis to the right half-plane. Before giving main theorem, we need to give some necessary lemmas.

**Lemma 3.1.** *Suppose that the $2i\pi$ periodic function $g$ is analytic in $\mathbb{C}_-$, all of its derivatives are continuous in $\overline{\mathbb{C}}_-$, and*

$$\sup_{z\in P}|g^{(k)}(z)|\leq\eta_k, \quad k\in\mathbb{N}_0.$$

*If the set $G\subset\left[-i\frac{\pi}{2},i\frac{3\pi}{2}\right]$ with*

*Lebesgue measure zero is the set of all zeros of the function $g$ with infinity multiplicity in $P$, and if*

$$\int_0^w \ln t(s)\mathrm{d}\mu(G_s)=-\infty,$$

*where* $t(s) = \inf_{k\in\mathbb{N}_0} \frac{\eta_k s^k}{k!}$ *and* $\mu(G_s)$ *is the Lebesgue measure of the* $s$*-neighborhood of* $G$*, and* $w > 0$ *is an arbitrary constant, then* $g \equiv 0$ *in* $\overline{\mathbb{C}}_-$ *(Bairamov et al. (2001)).*

**Lemma 3.2.**

$$|F^{(k)}(z)| \le \eta_k, \quad z \in P, \quad k \in \mathbb{N}_0 \tag{27}$$

*holds, where*

$$\eta_k \le Dd^k k! k^{k\left(\frac{1}{\delta}-1\right)}, \tag{28}$$

*and* $D$ *and* $d$ *are positive constants depending on* $C$*,* $\varepsilon$ *and* $\delta$*.*

*Proof.* Using (5) and (24), we obtain

$$|A(t,r)| \le C \exp\left(-\varepsilon\left(\frac{\ln r}{2\ln q}\right)^{\delta}\right), \quad t \in \{1,q\}, \quad r \in q^{\mathbb{N}}. \tag{29}$$

It follows from (14) and (29) that

$$|F^{(k)}(z)| \le C4^k + D_k, \quad z \in P, \quad k \in \mathbb{N}_0,$$

where

$$D_k = C4^k \sum_{r\in q^{\mathbb{N}}} \left(\frac{\ln r}{\ln q}\right)^k e^{-\frac{\varepsilon}{4}\left(\frac{\ln r}{\ln q}\right)^{\delta}}, \quad k \in q^{\mathbb{N}_0}.$$

We can also write for $D_k$

$$\begin{aligned} D_k &= C4^k \sum_{m=1}^{\infty} m^k e^{-\frac{\varepsilon}{4}m^{\delta}} \\ &= C4^k \int_0^n t^k e^{-\frac{\varepsilon}{4}t^{\delta}} \mathrm{d}t \le C4^k \int_0^{\infty} t^k e^{-\frac{\varepsilon}{4}t^{\delta}} \mathrm{d}t. \end{aligned}$$

If we define $y = \frac{\varepsilon}{4}t^{\delta}$, then we get

$$D_k \le C4^k \left(\frac{4}{\varepsilon}\right)^{\frac{k+1}{\delta}} \frac{1}{\delta} \int_0^{\infty} y^{\frac{k+1}{\delta}-1} e^{-y} \mathrm{d}y,$$

and using the definition and properties of Gamma function, we obtain

$$D_k \le C4^{2k+1} \left(\frac{4}{\varepsilon}\right)^{\frac{k+1}{\delta}} (k+1)^{\frac{1}{\delta}-1} (k+1)^{\frac{k}{\delta}}. \tag{30}$$

Using (30) and the inequalities $\left(1+\frac{1}{k}\right)^{\frac{k}{\delta}} < e^{\frac{1}{\delta}}$, $(k+1)^{\frac{1}{\delta}-1} < e^{\frac{k}{\delta}}$, and $k^k < k!e^k$, we have

$$D_k \leq Dd^k k! k^{k(\frac{1}{\delta}-1)}, \quad k \in \mathbb{N},$$

where $D$ and $d$ are positive constants depending on $\varepsilon$ and $\delta$. □

**Lemma 3.3.** *Assume* (24)*. Then* $M_4 = \emptyset$*.*

*Proof.* Using the Lemma 3.6, we find

$$\int_0^w \ln t(s) \mathrm{d}\mu(M_4, s) > -\infty, \tag{31}$$

where $t(s) = \inf_{k \in \mathbb{N}_0} \frac{\eta_k s^k}{k!}$, and $\mu(M_4, s)$ is the Lebesgue measure of $s$-neighborhood of $M_4$, and $\eta_k$ is defined by (28). Substituting (28) in the definition of $t(s)$, we find

$$t(s) = D \exp\left\{-\frac{1-\delta}{\delta} e^{-1} (ds)^{\frac{-\delta}{1-\delta}}\right\}. \tag{32}$$

Using (31) and (32), we can write

$$\int_0^w s^{-\frac{\delta}{1-\delta}} \mathrm{d}\mu(M_4, s) < \infty.$$

The last inequality holds for arbitrary $s$ if and only if $\mu(M_4, s) = 0$. This gives $M_4 = \emptyset$. □

Now, we can give the main theorem by using the last three lemmas.

**Theorem 3.4.** *Under the assumption* (24)*, the BVP* (1)–(2) *has a finite number of eigenvalues and spectral singularities with finite multiplicities.*

*Proof.* To prove the theorem we have to show that the function $F$ has a finite number of zeros with finite multiplicities in $P$. It follows from Theorem 3.3 and Lemma 3.3 that $M_3 = \emptyset$. So the bounded sets $M_1$ and $M_2$ have no limit points. This gives that the function $F$ has only finite number of zeros in $P$. Since $M_4 = \emptyset$, these zeros are of finite multiplicity. □

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